

# A family of warped product semi-Riemannian Einstein metrics

**Márcio Lemes de Sousa**

*ICET - CUA, Universidade Federal de Mato Grosso,*

*Av. Universitária nº 3.500, Pontal do Araguaia, MT, Brazil*

*e-mail: marciolemesew@yahoo.com.br*

**Romildo Pina \***

*IME, Universidade Federal de Goiás,*

*Caixa Postal 131, 74001-970, Goiânia, GO, Brazil*

*e-mail: romildo@ufg.br*

## Abstract

We study warped products semi-Riemannian Einstein manifolds. We consider the case in that the base is conformal to an  $n$ -dimensional pseudo-Euclidean space and invariant under the action of an  $(n - 1)$ -dimensional translation group. We provide all such solutions in the case Ricci-flat when the base is conformal to an  $n$ -dimensional pseudo-Euclidean space, invariant under the action of an  $(n - 1)$ -dimensional translation group and the fiber  $F$  is Ricci-flat. In particular, we obtain explicit solutions, in the case vacuum, for the Einstein field equation.

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## 1 Introduction and main statements

A semi-Riemannian manifold  $(M, g)$  is Einstein if there exists a real constant  $\lambda$  such that

$$Ric_g(X, Y) = \lambda g(X, Y)$$

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for each  $X, Y$  in  $T_p M$  and each  $p$  in  $M$ . This notion is relevant only for  $n \geq 4$ . Indeed, if  $n = 1$ ,  $Ric_g = 0$ . If  $n = 2$ , then at each  $p$  in  $M$ , we have  $Ric_g(X, Y) = \frac{1}{2}Kg(X, Y)$ , so a 2-dimensional semi-Riemannian manifold is Einstein if and only if it has constant sectional or scalar curvature. If  $n = 3$ , then  $(M, g)$  is Einstein if and only if it has constant sectional curvature.

Over the last few years, several authors have considered the following problem:

Let  $(M, g)$  be a semi-Riemannian manifold of dimension  $n > 3$ . Does there exist a metric  $g'$  on  $M$  such that  $(M, g')$  is an Einstein manifold?

According to T. Aubin [1], to decide if a Riemannian manifold carries an Einstein metric, will be one of the important questions in Riemannian geometry for the next decades. Finding solutions to this problem is equivalent to solving a nonlinear system of second-order partial differential equations. In particular, semi-Riemannian Einstein manifolds with zero Ricci curvature are special solutions, in the vacuum case ( $T = 0$ ), of following equation

$$Ric_g - \frac{1}{2}Kg = T, \quad (1)$$

where  $K$  the scalar curvature of  $g$  and  $T$  is a symmetric tensor of order 2. If  $g$  is the Lorentz metric on a four-dimensional manifold, this is simply the Einstein field equation. Whenever the tensor  $T$  represents a physical field such as electromagnetic field perfect fluid type, pure radiation field and vacuum ( $T = 0$ ), the above equation has been studied in several papers, most of them dealing with solutions which are invariant under some symmetry group of the equation (see [13] for details). When the metric  $g$  is conformal to the Minkowski space-time, then the solutions in the vacuum case are necessarily flat (see [13]).

Several authors constructed new examples of Einstein manifolds. In [15], Ziller constructed examples of compact manifolds with constant Ricci curvature. Chen, in [6], constructed new examples of Einstein manifolds with odd size, and, in [16], Yau presented a survey on Ricci-flat manifold. In [10], Kühnelt studied conformal transformations between Einstein spaces and, as a consequence of the obtained results, showed that there is no Riemannian Einstein Manifold with non-constant sectional curvature which is locally conformally flat. This result was extended to semi-Riemannian manifolds (see [12]). Accordingly to construct examples of Einstein manifolds with non-constant sectional curvature, we work with manifolds that are not locally conformally flat. A chance to build these manifolds is to work with warped product manifolds. Then considering  $(B, g_B)$  and  $(F, g_F)$  semi-Riemannian manifolds, and let  $f > 0$  be a smooth function on  $B$ , the warped

product  $M = B \times_f F$  is the product manifold  $B \times F$  furnished with the metric tensor

$$\tilde{g} = g_B + f^2 g_F,$$

$B$  is called the base of  $M = B \times_f F$ ,  $F$  is the fiber and  $f$  is the warping function. For example, polar coordinates determine a warped product in the case of constant curvature spaces, the case corresponds to  $\mathbb{R}^+ \times_r S^{n-1}$ .

There are several studies correlating warped product manifolds and locally conformally flat manifolds, see [3], [4], [5] and their references.

In a series of papers, the authors studied warped product Einstein manifolds under various conditions on the curvature and symmetry, see [7], [8] and [9]. Particularly, He-Petersen-Wyle, [8], characterized warped product Riemannian Einstein metrics when the base is locally conformally flat.

It is well-known that the Einstein condition on warped geometries requires that the fibers must be necessarily Einstein (see [2]). In this paper, initially we give a characterization for warped product semi-Riemannian Einstein manifold when the base is locally conformally flat. Using this characterization, we present new examples of semi-Riemannian manifolds with zero Ricci curvature. More precisely, let us consider  $(\mathbb{R}^n, g)$  the pseudo-Euclidean space,  $n \geq 3$ , with coordinates  $x = (x_1, \dots, x_n)$ ,  $g_{ij} = \delta_{ij} \varepsilon_i$  and  $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$  a warped product, where  $\bar{g} = \frac{1}{\varphi^2} g$ ,  $F$  is a semi-Riemannian Einstein manifold with constant Ricci curvature  $\lambda_F$ ,  $m \geq 1$ ,  $f, \varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  are smooth functions, where  $f$  is a positive function. In Theorem 1.1, we find necessary and sufficient conditions for the warped product metric  $\tilde{g} = \bar{g} + f^2 g_F$  to be Einstein. In Theorem 1.2, we consider  $f$  and  $\varphi$  invariant under the action of an  $(n-1)$ -dimensional translation group and let  $\xi = \sum_{i=1}^n \alpha_i x_i$ ,  $\alpha_i \in \mathbb{R}$ , be a basic invariant for an  $(n-1)$ -dimensional translation group. We want to obtain differentiable functions  $\varphi(\xi)$  and  $f(\xi)$ , such that the metric  $\tilde{g}$  is Einstein. We first obtain necessary and sufficient conditions on  $f(\xi)$  and  $\varphi(\xi)$  for the existence of  $\tilde{g}$ . We show that these conditions are different depending on the direction  $\alpha = \sum_{i=1}^n \alpha_i \partial / \partial x_i$  being null (lightlike) or not. We observe that, in the null case, the metrics  $\tilde{g}$  and  $g_F$  are necessarily Ricci-flat.

Considering  $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$  and  $F$  a Ricci-flat manifold, we obtain all the metrics  $\tilde{g} = \bar{g} + f^2 g_F$  which are Ricci-flat and invariant under the action of an  $(n-1)$ -dimensional translation group. We prove that, if the direction  $\alpha$  is timelike or spacelike, the functions  $f$  and  $\varphi$  depend on the dimensions  $n, m$  and also on a finite number of parameters. In fact, the solutions are explicitly given in Theorems 1.3 and 1.4. If the direction  $\alpha$  is null, then

there are infinitely many solutions. In fact, in this case, for any given positive differentiable function  $f(\xi)$ , the function  $\varphi(\xi)$  satisfies a linear ordinary differential equation of second order (see Theorem 1.5). We illustrate this fact with some explicit examples. The metrics obtained in Theorems 1.3, 1.4 and 1.5 are explicit solutions, in the vacuum case, of the equation (1). Especially considering  $\tilde{g}$  Lorentz,  $n = 3$  and  $m = 1$  in Theorems 1.3 and 1.5, we obtain explicit solutions, in the case vacuum, for the Einstein field equation.

**Remark 1.1** *When the dimension of the fiber  $F$  is  $m = 1$ , we consider  $M = (\mathbb{R}^n, \bar{g}) \times_f \mathbb{R}$  and, in this case,  $\lambda_F = 0$ .*

In what follows, we state our main results. We denote by  $\varphi_{,x_i x_j}$  and  $f_{,x_i x_j}$  the second order derivatives of  $\varphi$  and  $f$  with respect to  $x_i x_j$ .

**Theorem 1.1** *Let  $(\mathbb{R}^n, g)$  be a pseudo-Euclidean space,  $n \geq 3$ , with coordinates  $x = (x_1, \dots, x_n)$  and  $g_{ij} = \delta_{ij} \varepsilon_i$ . Consider a warped product  $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$ , where  $\bar{g} = \frac{1}{\varphi^2} g$ ,  $F$  is a semi-Riemannian Einstein manifold with constant Ricci curvature  $\lambda_F$ ,  $m \geq 1$ ,  $f, \varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  are smooth functions, where  $f$  is a positive function. Then the warped product metric  $\tilde{g} = \bar{g} + f^2 g_F$  is Einstein with constant Ricci curvature  $\lambda$  if, and only if, the functions  $f$  and  $\varphi$  satisfy:*

$$(n-2)f\varphi_{,x_i x_j} - m\varphi f_{,x_i x_j} - m\varphi_{,x_i} f_{,x_j} - m\varphi_{,x_j} f_{,x_i} = 0, \quad i \neq j, \quad (2)$$

$$\begin{aligned} \varphi[(n-2)f\varphi_{,x_i x_i} - m\varphi f_{,x_i x_i} - 2m\varphi_{,x_i} f_{,x_i}] &+ \varepsilon_i \left[ f\varphi \sum_{k=1}^n \varepsilon_k \varphi_{,x_k x_k} \right. \\ &\left. - (n-1)f \sum_{k=1}^n \varepsilon_k \varphi_{,x_k}^2 + m\varphi \sum_{k=1}^n \varepsilon_k \varphi_{,x_k} f_{,x_k} \right] \\ &= \varepsilon_i \lambda f, \end{aligned} \quad (3)$$

and

$$-f\varphi^2 \sum_{k=1}^n \varepsilon_k f_{,x_k x_k} + (n-2)f\varphi \sum_{k=1}^n \varepsilon_k f_{,x_k} \varphi_{,x_k} - (m-1)\varphi^2 \sum_{k=1}^n \varepsilon_k f_{,x_k}^2 = \lambda f^2 - \lambda_F. \quad (4)$$

We want to find solutions of the system (2), (3) and (4) of the form  $\varphi(\xi)$  and  $f(\xi)$ , where  $\xi = \sum_{i=1}^n \alpha_i x_i$ ,  $\alpha_i \in \mathbb{R}$ . Whenever  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 \neq 0$ , without loss of generality, we may consider  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = \pm 1$ . The following theorem provides the system of ordinary differential equations that must be satisfied by such solutions.

**Theorem 1.2.** Let  $(\mathbb{R}^n, g)$  be a pseudo-Euclidean space,  $n \geq 3$ , with coordinates  $x = (x_1, \dots, x_n)$  and  $g_{ij} = \delta_{ij}\varepsilon_i$ . Consider  $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$ , where  $\bar{g} = \frac{1}{\varphi^2}g$ ,  $F^m$  a semi-Riemannian Einstein manifold with constant Ricci curvature  $\lambda_F$  and smooth functions  $\varphi(\xi)$  and  $f(\xi)$ , where  $\xi = \sum_{i=1}^n \alpha_i x_i$ ,  $\alpha_i \in \mathbb{R}$ , and  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = \varepsilon_{i_0}$  or  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = 0$ . Then  $M$  is Einstein with constant Ricci curvature  $\lambda$  if, and only if, the functions  $f$  and  $\varphi$  satisfy:

$$(i) \quad \begin{cases} (n-2)f\varphi'' - m\varphi f'' - 2m\varphi' f' = 0, \\ \sum_{k=1}^n \varepsilon_k \alpha_k^2 [f\varphi\varphi'' - (n-1)f\varphi'^2 + m\varphi\varphi' f'] = \lambda f \\ \sum_{k=1}^n \varepsilon_k \alpha_k^2 [-f\varphi^2 f'' + (n-2)f\varphi\varphi' f' - (m-1)\varphi^2 f'^2] = \lambda f^2 - \lambda_F, \end{cases} \quad (5)$$

whenever  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = \varepsilon_{i_0}$ , and

$$(ii) \quad (n-2)f\varphi'' - m\varphi f'' - 2m\varphi' f' = 0, \quad \text{and} \quad \lambda = \lambda_F = 0, \quad (6)$$

whenever  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = 0$ ,

In the following result we provide all the solutions of (5) when  $m = 1$  and  $M$  is Ricci-flat.

**Theorem 1.3.** Let  $(\mathbb{R}^n, g)$  be a pseudo-Euclidean space,  $n \geq 3$ , with coordinates  $x = (x_1, \dots, x_n)$  and  $g_{ij} = \delta_{ij}\varepsilon_i$ . Consider non-constant smooth functions  $\varphi(\xi)$  and  $f(\xi)$ , where  $\xi = \sum_{i=1}^n \alpha_i x_i$ ,  $\alpha_i \in \mathbb{R}$ , and  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = \varepsilon_{i_0}$ . Then the warped product  $M = (\mathbb{R}^n, \bar{g}) \times_f \mathbb{R}$ , with  $\bar{g} = \frac{1}{\varphi^2}g$ , is a Ricci-flat manifold if, and only if,

$$\begin{cases} \varphi(\xi) = \left[ \frac{2}{(-n+2)k_1\xi + k_2} \right]^{\frac{2}{n-2}} \\ f(\xi) = \frac{2k}{(-n+2)k_1\xi + k_2} \end{cases} \quad (7)$$

where  $k, k_1$  and  $k_2$  are constant with  $k, k_1 > 0$ . These solutions are defined on the half space determined by  $\sum_{i=1}^n \alpha_i x_i < \frac{k_2}{(n-2)k_1}$ .

**Theorem 1.4** Let  $(\mathbb{R}^n, g)$  be a pseudo-Euclidean space,  $n \geq 3$ , with coordinates  $x = (x_1, \dots, x_n)$  and  $g_{ij} = \delta_{ij}\varepsilon_i$ . Consider non-constant smooth functions  $\varphi(\xi)$  and  $f(\xi)$ , where

$\xi = \sum_{i=1}^n \alpha_i x_i$ ,  $\alpha_i \in \mathbb{R}$ ,  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = \varepsilon_{i_0}$ , and the warped product  $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$ , with  $\bar{g} = \frac{1}{\varphi^2} g$  and  $F^m$  is a Ricci-flat manifold with  $m \geq 2$ . Then  $M$  is a Ricci-flat manifold if, and only if,

$$\begin{cases} \varphi_{\mp}(\xi) &= k [\mp \beta (k_1 \xi + k_2)]^{\mp \frac{\alpha}{\beta}} \\ f_{\mp}(\xi) &= [\mp \beta (k_1 \xi + k_2)]^{\mp \frac{1}{\beta}} \end{cases} \quad (8)$$

where  $k, k_1, k_2, \alpha, \beta$  are constants with,  $k, k_1 > 0$ ,  $\beta = \frac{\sqrt{m(n-1)(m+n-2)}}{n-1}$  and  $\alpha = \frac{m \pm \beta}{n-2}$ . The solutions  $\varphi_-$  and  $f_-$  are defined on the half space determined by  $\sum_{i=1}^n \alpha_i x_i < -\frac{k_2}{k_1}$ , while that  $\varphi_+$  and  $f_+$  are defined on the half space determined by  $\sum_{i=1}^n \alpha_i x_i > -\frac{k_2}{k_1}$ .

The following theorem shows that there are infinitely many of warped products  $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$  Ricci-flat, which are invariant under the action of an  $(n-1)$ -dimensional group acting on  $\mathbb{R}^n$ , when  $\alpha = \sum_{i=1}^n \alpha_i \partial / \partial x_i$  is null-like vector.

**Theorem 1.5** Let  $f(\xi)$  be any positive differentiable function, where  $\xi = \sum_{i=1}^n \alpha_i x_i$  and  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = 0$ . Then there exists a function  $\varphi(\xi)$  satisfying (6) and  $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$  is a Ricci-flat manifold.

Before proving our main results, we give an example illustrating Theorem 1.5. Let  $f(\xi) = ke^{A\xi}$  where  $k > 0$  and  $A \neq 0$ ,  $\xi = \sum_{i=1}^n \alpha_i x_i$  and  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = 0$ . Solving (6) we get

$$\varphi(\xi) = c_1 e^{A \left( \frac{m + \sqrt{m(n-1)}}{n-2} \right) \xi} + c_2 e^{A \left( \frac{m - \sqrt{m(n-1)}}{n-2} \right) \xi}, \quad c_1, c_2 \in \mathbb{R}.$$

It follows from Theorem 1.5 that  $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$  is a Ricci-flat manifold. Considering  $c_1$  and  $c_2$  positive constants we have that  $\varphi$  is globally defined on  $\mathbb{R}^n$ .

**Remark 1.2** The condition that the warped product of an  $n$ -dimensional Riemannian manifold  $(M^n, g)$  with a second  $m$ -dimensional Einstein Riemannian manifold to be again Einstein can be expressed as the Einstein condition

$$\text{Ric}_f^m = \text{Ric}_g + \text{Hess}_g f - \frac{1}{m} df \otimes df = \lambda g \quad (9)$$

for the Bakry-Emery Ricci tensor of  $g$  and the warping function  $f \in C^\infty(M)$ . The Riemannian metric  $g$  and a function  $f$  satisfying the above condition is called  $(\lambda, n+m)$ -Einstein metric. The  $(\lambda, n+1)$ -Einstein metrics are more commonly called static metrics and such metrics have been extensively studied for their connections to scalar curvature, the positive mass theorem and general relativity. For more details see [7].

In this paper, in the Theorems 1.3, 1.4 and 1.5, we construct explicit examples of solutions of the equation (9) with  $\lambda = 0$  in the semi-Riemannian case.

## 2 Proofs of the Main Results

**Proof of Theorem 1.1:** Assume initially that  $m > 1$ . It follows from [14] that if  $X_1, X_2, \dots, X_n \in \mathcal{L}(\mathbb{R}^n)$  and  $Y_1, Y_2, \dots, Y_m \in \mathcal{L}(F)$  ( $\mathcal{L}(\mathbb{R}^n)$  and  $\mathcal{L}(F)$  are respectively the lift of a vector field on  $\mathbb{R}^n$  and  $F$  to  $\mathbb{R}^n \times F$ ), then

$$\begin{cases} Ric_{\bar{g}}(X_i, X_j) &= Ric_{\bar{g}}(X_i, X_j) - \frac{m}{f} Hess_{\bar{g}}f(X_i, X_j), \quad \forall i, j = 1, \dots, n \\ Ric_{\bar{g}}(X_i, Y_j) &= 0, \quad \forall i = 1, \dots, n, j = 1, \dots, m \\ Ric_{\bar{g}}(Y_i, Y_j) &= Ric_{g_F}(Y_i, Y_j) - \tilde{g}(Y_i, Y_j) \left( \frac{\Delta_{\bar{g}} f}{f} + (m-1) \frac{\tilde{g}(\nabla f, \nabla f)}{f^2} \right), \quad \forall i, j = 1, \dots, m \end{cases} \quad (10)$$

It is well known (see, ex. [2]) that if  $\bar{g} = \frac{1}{\varphi^2} g$ , then

$$Ric_{\bar{g}} = \frac{1}{\varphi^2} \{ (n-2) \varphi Hess_g(\varphi) + [\varphi \Delta_g \varphi - (n-1) |\nabla_g \varphi|^2] g \}.$$

Since  $g(X_i, X_j) = \varepsilon_i \delta_{ij}$ , we have

$$\begin{aligned} Ric_{\bar{g}}(X_i, X_j) &= \frac{1}{\varphi} \{ (n-2) Hess_g(\varphi)(X_i, X_j) \} \quad \forall i \neq j = 1, \dots, n \\ Ric_{\bar{g}}(X_i, X_i) &= \frac{1}{\varphi^2} \{ (n-2) \varphi Hess_g(\varphi)(X_i, X_i) + [\varphi \Delta_g \varphi - (n-1) |\nabla_g \varphi|^2] \varepsilon_i \} \quad \forall i = 1, \dots, n. \end{aligned}$$

As  $Hess_g(\varphi)(X_i, X_j) = \varphi_{,x_i x_j}$ ,  $\Delta_g \varphi = \sum_{k=1}^n \varepsilon_k \varphi_{,x_k x_k}$  and  $|\nabla_g \varphi|^2 = \sum_{k=1}^n \varepsilon_k \varphi_{,x_k}^2$ , we have

$$\begin{cases} Ric_{\bar{g}}(X_i, X_j) &= \frac{(n-2) \varphi_{,x_i x_j}}{\varphi} \quad \forall i \neq j : 1 \dots n \\ Ric_{\bar{g}}(X_i, X_i) &= \frac{(n-2) \varphi_{,x_i x_i} + \varepsilon_i \sum_{k=1}^n \varepsilon_k \varphi_{,x_k x_k}}{\varphi} - (n-1) \varepsilon_i \sum_{k=1}^n \frac{\varepsilon_k \varphi_{,x_k}^2}{\varphi^2} \end{cases} \quad (11)$$

Recall that

$$Hess_{\bar{g}}(f)(X_i, X_j) = f_{,x_i x_j} - \sum_k \bar{\Gamma}_{ij}^k f_{,x_k},$$

where  $\bar{\Gamma}_{ij}^k$  are the Christoffel symbols of the metric  $\bar{g}$ . For  $i, j, k$  distincts, we have

$$\bar{\Gamma}_{ij}^k = 0 \quad \bar{\Gamma}_{ij}^i = -\frac{\varphi_{,x_j}}{\varphi} \quad \bar{\Gamma}_{ii}^k = \varepsilon_i \varepsilon_k \frac{\varphi_{,x_k}}{\varphi} \quad \bar{\Gamma}_{ii}^i = -\frac{\varphi_{,x_i}}{\varphi}$$

therefore,

$$\begin{cases} Hess_{\bar{g}}(f)(X_i, X_j) &= f_{,x_i x_j} + \frac{\varphi_{,x_j}}{\varphi} f_{,x_i} + \frac{\varphi_{,x_i}}{\varphi} f_{,x_j}, \quad \forall i \neq j = 1 \dots n \\ Hess_{\bar{g}}(f)(X_i, X_i) &= f_{,x_i x_i} + 2 \frac{\varphi_{,x_i}}{\varphi} f_{,x_i} - \varepsilon_i \sum_{k=1}^n \varepsilon_k \frac{\varphi_{,x_k}}{\varphi} f_{,x_k} \end{cases} \quad (12)$$

Substituting (11) and (12) in the first equation of (10), we obtain

$$Ric_{\bar{g}}(X_i, X_j) = \frac{(n-2)\varphi_{,x_i x_j}}{\varphi} - \frac{m}{f} \left[ f_{,x_i x_j} + \frac{\varphi_{,x_j}}{\varphi} f_{,x_i} + \frac{\varphi_{,x_i}}{\varphi} f_{,x_j} \right], \quad \forall i \neq j \quad (13)$$

and

$$\begin{aligned} Ric_{\bar{g}}(X_i, X_i) &= \frac{(n-2)\varphi_{,x_i x_i} + \varepsilon_i \sum_{k=1}^n \varepsilon_k \varphi_{,x_k x_k}}{\varphi} - (n-1)\varepsilon_i \sum_{k=1}^n \frac{\varepsilon_k \varphi_{,x_k}^2}{\varphi^2} \\ &\quad - \frac{m}{f} \left[ f_{,x_i x_i} + 2 \frac{\varphi_{,x_i}}{\varphi} f_{,x_i} - \varepsilon_i \sum_{k=1}^n \varepsilon_k \frac{\varphi_{,x_k}}{\varphi} f_{,x_k} \right]. \end{aligned} \quad (14)$$

On the other hand,

$$\begin{cases} Ric_{g_F}(Y_i, Y_j) &= \lambda_F g_F(Y_i, Y_j) \\ \tilde{g}(Y_i, Y_j) &= f^2 g_F(Y_i, Y_j) \\ \Delta_{\bar{g}} f &= \varphi^2 \sum_{k=1}^n \varepsilon_k f_{,x_k x_k} - (n-2)\varphi \sum_{k=1}^n \varepsilon_k \varphi_{,x_k} f_{,x_k} \\ \tilde{g}(\nabla f, \nabla f) &= \varphi^2 \sum_{k=1}^n \varepsilon_k f_{,x_k}^2 \end{cases} \quad (15)$$

Substituting (15) in the third equation of the system (10), we have

$$Ric_{\tilde{g}}(Y_i, Y_j) = \gamma_{ij} g_F(Y_i, Y_j) \quad (16)$$

where,

$$\gamma_{ij} = \lambda_F - f\varphi^2 \sum_{k=1}^n \varepsilon_k f_{,x_k x_k} + (n-2)f\varphi \sum_{k=1}^n \varepsilon_k \varphi_{,x_k} f_{,x_k} - (m-1)\varphi^2 \sum_{k=1}^n \varepsilon_k f_{,x_k}^2.$$

Using the equations (13), (14), (16) and the second equation of (10), we have  $(M, \tilde{g})$  is an Einstein manifold if, and only if, the equations (2), (3) and (4) are satisfied. In the case  $m = 1$  just remember that:

$$\begin{aligned} Ric_{\tilde{g}}(X_i, X_j) &= Ric_{\bar{g}}(X_i, X_j) - \frac{1}{f} Hess_{\bar{g}} f(X_i, X_j), \quad \forall i, j = 1, \dots, n \\ Ric_{\tilde{g}}(X_i, Y) &= 0, \quad \forall i = 1, \dots, n \\ Ric_{\tilde{g}}(Y, Y) &= -\tilde{g}(Y, Y) \frac{\Delta_{\bar{g}} f}{f}. \end{aligned}$$



In this case, the equation (2) and (3) remain the same and the equation (4) reduces to

$$-f\varphi^2 \sum_{k=1}^n \varepsilon_k f_{,x_k x_k} + (n-2)f\varphi \sum_{k=1}^n \varepsilon_k f_{,x_k} \varphi_{,x_k} = \lambda f^2.$$

This concludes the proof of Theorem 1.1. □

**Proof of Theorem 1.2** Assume initially that  $m > 1$ . Let  $\bar{g} = \varphi^{-2}g$  be a conformal metric of  $g$ . We are assuming that  $\varphi(\xi)$  and  $f(\xi)$  are functions of  $\xi$ , where  $\xi = \sum_{i=1}^n \alpha_i x_i$ ,  $\alpha_i \in \mathbb{R}$  and  $\sum_i \varepsilon_i \alpha_i^2 = \varepsilon_{i_0}$  or  $\sum_i \varepsilon_i \alpha_i^2 = 0$ . Hence, we have

$$\varphi_{,x_i} = \varphi' \alpha_i, \quad \varphi_{,x_i x_j} = \varphi'' \alpha_i \alpha_j$$

and

$$f_{,x_i} = f' \alpha_i, \quad f_{,x_i x_j} = f'' \alpha_i \alpha_j.$$

Substituting these expressions into (2), we get

$$(n-2)f\varphi'' \alpha_i \alpha_j - m\varphi f'' \alpha_i \alpha_j - 2m\varphi' f' \alpha_i \alpha_j = 0, \quad \forall i \neq j.$$

If there exist  $i \neq j$  such that  $\alpha_i \alpha_j \neq 0$ , then this equation reduces to

$$(n-2)f\varphi'' - m\varphi f'' - 2m\varphi' f' = 0. \quad (17)$$

Similarly, considering equation (3), we get

$$\alpha_i^2 \varphi [(n-2)f\varphi'' - m\varphi f'' - 2m\varphi' f'] + \varepsilon_i \sum_k \varepsilon_k \alpha_k^2 [f\varphi\varphi'' - (n-1)f(\varphi')^2 + m\varphi\varphi' f'] = \varepsilon_i \lambda f.$$

Due to the relation between  $\varphi''$  and  $f''$  given in (17), the above equation reduces to

$$\sum_k \varepsilon_k \alpha_k^2 [f\varphi\varphi'' - (n-1)f(\varphi')^2 + m\varphi\varphi' f'] = \lambda f. \quad (18)$$

Analogously, the equation (4) reduces to

$$\sum_k \varepsilon_k \alpha_k^2 [-f\varphi^2 f'' + (n-2)f\varphi\varphi' f' - (m-1)\varphi^2 f'^2] = \lambda f^2 - \lambda_F. \quad (19)$$

Therefore, if  $\sum_k \varepsilon_k \alpha_k^2 = \varepsilon_{i_0}$ , we obtain the equations of the system (5). If  $\sum_k \varepsilon_k \alpha_k^2 = 0$ , we have (17) satisfied and (18) implies  $\lambda = 0$ , hence  $\lambda_F = 0$ , i.e., (6) holds.

If for all  $i \neq j$ , we have  $\alpha_i \alpha_j = 0$ , then  $\xi = x_{i_0}$  and equation (3) is trivially satisfied for all  $i \neq j$ . Considering (2) for  $i \neq i_0$ , we get

$$\sum_{k=1}^n \varepsilon_k \alpha_k^2 [f\varphi\varphi'' - (n-1)f\varphi'^2 + m\varphi\varphi' f'] = \lambda f$$

and hence, the second equation of (5) is satisfied. Considering  $i = i_0$  in (2) we get that the first equation of (5) is satisfied.

Considering  $i = i_0$  or  $i \neq i_0$  in (4), we get that the third equation of (5) is satisfied.

When  $m = 1$ , the first and the second equation of the system (5) are the same and the third equation reduces to

$$\sum_{k=1}^n \varepsilon_k \alpha_k^2 [-f\varphi^2 f'' + (n-2)f\varphi\varphi' f'] = \lambda f^2.$$

This concludes the proof of Theorem 1.2. □

**Proof of Theorem 1.3.** We consider smooth functions  $\varphi(\xi)$  and  $f(\xi)$ , where  $\xi = \sum_{i=1}^n \alpha_i x_i$ ,  $\alpha_i \in \mathbb{R}$ ,  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = \pm 1$ . It follows from Theorem 1.2 that the metric  $\tilde{g} = \bar{g} + f^2 g_F$  is Ricci-flat if, and only if,  $\varphi$  and  $f$  satisfy

$$\begin{cases} (n-2)f\varphi'' - \varphi f'' - 2\varphi' f' &= 0 \\ f\varphi\varphi'' - (n-1)f(\varphi')^2 + \varphi\varphi' f' &= 0 \\ \varphi f'' &= (n-2)\varphi' f'. \end{cases} \quad (20)$$

By substituting  $\varphi f''$  in the first equation of (20), we get

$$f\varphi'' = \frac{n}{n-2}\varphi' f'. \quad (21)$$

Hence substituting equation (21) in the second equation of (20), we have

$$\frac{f'}{f} = \frac{(n-2)}{2} \frac{\varphi'}{\varphi}$$

hence we get,

$$f = k\varphi^{\frac{n-2}{2}}, \quad (22)$$

where  $k$  is a positive constant. By deriving the equation (22), we obtain

$$f' = \frac{k(n-2)}{2} \varphi^{\frac{n-4}{2}} \varphi'$$

and

$$f'' = \frac{k(n-2)(n-4)}{4} \varphi^{\frac{n-6}{2}} (\varphi')^2 + \frac{k(n-2)}{2} \varphi^{\frac{n-4}{2}} \varphi''.$$

Substituting  $f$ ,  $f'$  e  $f''$  in the first equation of (20), we get

$$\frac{\varphi''}{\varphi'} = \frac{n}{2} \frac{\varphi'}{\varphi}$$

hence we have

$$\varphi(\xi) = \left[ \frac{2}{(-n+2)(k_1\xi + k_2)} \right]^{\frac{2}{n-2}}, \quad (23)$$

where  $k_1, k_2$  are constants with  $k_1 > 0$ . By substituting (23) in (22), we obtain

$$f(\xi) = \frac{2k}{(-n+2)(k_1\xi + k_2)}.$$

Finally, it is easily seen that  $\varphi$  and  $f$  satisfy the system (20).

This concludes the proof of Theorem 1.3. □

**Proof of Theorem 1.4.** We consider smooth functions  $\varphi(\xi)$  and  $f(\xi)$ , where  $\xi = \sum_{i=1}^n \alpha_i x_i$ ,  $\alpha_i \in \mathbb{R}$ ,  $\sum_{i=1}^n \epsilon_i \alpha_i^2 = \pm 1$ . It follows from Theorem 1.2 that the metric  $\tilde{g} = \bar{g} + f^2 g_F$ , is Ricci-flat if, and only if,  $\varphi$  and  $f$  satisfy

$$\begin{cases} (n-2)f\varphi'' - m\varphi f'' - 2m\varphi' f' = 0, \\ f\varphi\varphi'' - (n-1)f\varphi'^2 + m\varphi\varphi' f' = 0 \\ -f\varphi^2 f'' + (n-2)f\varphi\varphi' f' - (m-1)\varphi^2 f'^2 = 0. \end{cases} \quad (24)$$

The system (24) is equivalent to

$$\begin{cases} (n-2) \left( \frac{\varphi'}{\varphi} \right)' + (n-2) \left( \frac{\varphi'}{\varphi} \right)^2 - m \left( \frac{f'}{f} \right)' - m \left( \frac{f'}{f} \right)^2 - 2m \frac{\varphi'}{\varphi} \frac{f'}{f} = 0, \\ \left( \frac{\varphi'}{\varphi} \right)' - (n-2) \left( \frac{\varphi'}{\varphi} \right)^2 + m \frac{\varphi'}{\varphi} \frac{f'}{f} = 0 \\ \left( \frac{f'}{f} \right)' - (n-2) \frac{\varphi'}{\varphi} \frac{f'}{f} + m \left( \frac{f'}{f} \right)^2 = 0, \end{cases} \quad (25)$$

isolating  $\left( \frac{f'}{f} \right)'$  in the third equation of (25) we obtain,

$$\left( \frac{f'}{f} \right)' = (n-2) \frac{\varphi'}{\varphi} \frac{f'}{f} - m \left( \frac{f'}{f} \right)^2. \quad (26)$$

Substituting (26) in the first equation of (25) we have

$$\left( \frac{\varphi'}{\varphi} \right)' = \frac{mn}{n-2} \frac{\varphi'}{\varphi} \frac{f'}{f} - \left( \frac{\varphi'}{\varphi} \right)^2 - \frac{m(m-1)}{n-2} \left( \frac{f'}{f} \right)^2. \quad (27)$$

Substituting (27) in the second equation of (25) we get

$$(n-1) \left( \frac{\varphi'}{\varphi} \right)^2 - 2 \frac{m(n-1)}{n-2} \frac{\varphi'}{\varphi} \frac{f'}{f} + \frac{m(m-1)}{n-2} \left( \frac{f'}{f} \right)^2 = 0$$

hence we have

$$\frac{\varphi'}{\varphi} = \frac{m(n-1) \pm \sqrt{m(n-1)(m+n-2)}}{(n-1)(n-2)} \frac{f'}{f}.$$

Making  $\alpha = \frac{m(n-1) \pm \sqrt{m(n-1)(m+n-2)}}{(n-1)(n-2)}$ , we have

$$\varphi = k f^\alpha$$

where  $k > 0$  is a constant. By deriving this equation we obtain

$$\varphi' = k \alpha f^{\alpha-1} f'.$$

Substituting  $\varphi$  and  $\varphi'$  in the third equation of (25), we get

$$\frac{f''}{f} = [(n-2)\alpha - (m-1)] \left( \frac{f'}{f} \right)^2.$$

Note that  $(n-2)\alpha - (m-1) = 1 \pm \beta$ , where  $\beta = \frac{\sqrt{m(n-1)(m+n-2)}}{(n-1)}$ , hence we have

$$\frac{f''}{f'} = (1 \pm \beta) \frac{f'}{f}. \quad (28)$$

Integrating the equation (28) we obtain:

$$f(\xi) = [\mp \beta(k_1 \xi + k_2)]^{\mp \frac{1}{\beta}}, \quad (29)$$

with  $k_1, k_2$  constants and  $k_1 > 0$ , hence

$$\varphi(\xi) = [\mp \beta(k_1 \xi + k_2)]^{\mp \frac{\alpha}{\beta}}. \quad (30)$$

Finally, it is easily seen that  $\varphi$  and  $f$  given by (29) and (30) satisfy the system (24).

This concludes the proof of Theorem 1.4. □

**Proof of Theorem 1.5.** Let  $f(\xi)$  be any positive differentiable function invariant under translation of an given  $(n-1)$ -dimensional translation group, whose basic invariant  $\xi = \sum_{i=1}^n \alpha_i x_i$ , where  $\alpha_i \in \mathbb{R}$  and  $\sum_{i=1}^n \varepsilon_i \alpha_i^2 = 0$ . Then it follows from Theorem 1.2 that  $M = (\mathbb{R}^n, \bar{g}) \times_f F^m$  is a Ricci-flat manifold, with  $f$  being warping function, if, and only if,  $\lambda = \lambda_F = 0$  and  $\varphi$  satisfies the linear ordinary differential equation (6) determined by  $f$ . □

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